Prediction of Transient Temperature Distributions with Embedded Thermocouples

MURRAY IMBER* AND JAMAL KHAN†
Polytechnic Institute of Brooklyn, Brooklyn, New York

From theoretical considerations, an analytical method has been developed for the inverse heat conduction problem when the temperatures are known at two positions. Based upon these interior thermocouple readings, a closed form solution is obtained, via Laplace transform techniques, for the transient temperatures beyond the two positions. The method allows for the replacement of the imput thermocouple data by a temporal power series and a second series of error functions weighted by powers of time. The resultant expression for the prediction temperature is in the form of a summation of the repeated integrals of the error function. The method may be used to determine boundary conditions at either face of a finite slab or hollow sphere, with high accuracy. By direct application, temperature extrapolation is also feasible for multilayered mediums. Numerical examples are presented as verification of the method.

Nomenclature

A_m, a_n, B_m, b_n	= general coefficients
C	= integration constant
$i^n \operatorname{erfc} x$	= repeated error integral of variable x
K	= thermal conductivity
k	= $(\alpha_1/\alpha_2)^{1/2}$
n, m, q	= summation term
N	= number of equal parts
P	= $(s/\alpha)^{1/2}$
s	= Laplace transform variable
t	= time variable
T	= temperature
$\bar{T}$	= Laplace transform of temperature
T'	= temperature in region $x < \bar{x}$
T''	= temperature in region $x > \bar{x}$
x	= space variable
$\bar{x}$	= interface location between two materials
α	= thermal diffusivity
Δ	= distance between thermocouples, $x_2 - x_1$
σ	= kK_2/K_1

Subscripts

i	= inner face location
1	= property at first thermocouple
2	= property at second thermocouple
3	= property at third thermocouple

Introduction

IN most industrial applications, the direct solution of the conventional heat conduction equation offers little difficulty provided that the boundary conditions may be assessed with a fair degree of accuracy. Normally, this is accomplished experimentally via thermocouples positioned at the boundaries, and analysis is used to determine interior transient temperature behavior. Unfortunately, there are situations where the positioning of surface measurement instruments is precluded; consequently the thermocouples must be imbedded in the solid's interior. The problem, therefore, is to determine the surface behavior that produces the recorded internal temperature responses. This task is distinctly different from the direct conduction problem, as previously posed, and it is identified as the inverse heat conduction problem.

When data is available from several interior thermocouple positions, the natural inclination is to attempt an extrapolation to the surface by a curve fitting procedure. Clearly, this method is undesirable since, for large surface heat fluxes, the temperature gradients are significant in the vicinity of the boundary and the curve fit cannot reflect this behavior. There are, however, a number of advanced analytical methods which may be utilized to overcome this difficulty. A review of the literature indicates that these methods can generally be separated into two categories: solutions requiring a series formulation, and those methods applying transform techniques. In what follows, a brief review of the prominent methods will be presented.

Carslaw[1] suggested a solution to the heat conduction equation in the form of a Taylor series about an interior point. By substitution of the series into the conduction equation, the coefficients in the power series expansion is directly related to the higher order time derivatives of the local temperature and heat flux. The associated coefficients of the even-power terms in the series are the interior temperature and its higher derivatives, whereas the odd-power term coefficients are the local flux and its higher derivatives. Thus, the temperature for any other location, i.e., the prediction temperature, is determined from the thermal response at an interior point. Burggraf[2] achieved a generalization of the method to include cylinders and spheres. In this method, the solution is assumed, a priori, to be in the form of two series: the local temperature, or its higher order derivatives, multiplied by unknown spatial functions, and the local heat flux, or its higher order derivatives, multiplied by another set of unknown spatial functions. The functional forms of the spatial terms are determined by the differential equations that follow when each of the series is substituted into the conduction equation. For solid cylinders or spheres, the prediction temperature is a summation involving only the local temperature and its higher order derivatives. The hollow cylinder case is touched upon briefly, and it should be noted that the local temperature, heat

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* Professor, Mechanical Engineering Department; currently Visiting Professor, University of Windsor, Windsor, Ontario, Canada. Member AIAA.
† Graduate Student.

flux and their derivatives must all be utilized. Kover'yanov[3] though primarily concerned with the nonlinear inverse conduction problem, developed fully the series solution for the hollow cylinder. The prediction temperature is represented as an infinite series of separable products of time and space functions. From the conduction equation, differential equations are obtained for each of these terms which are, in turn, integrated. The method appears to be analogous to Burggraf's development, and the final form for the prediction temperature incorporates the higher order derivative terms previously mentioned.

Shumakov[4] developed a solution to the inverse conduction problem for a finite slab by reformulating it as a series of direct conduction problems. Initially, the surface heat flux variation is approximated by a function with step-wise changes in the heat flux, and solutions are obtained for each associated step change. The initial condition corresponding to each time interval is obtained from the temperature expressions pertaining to the preceding time interval. For values of the Fourier number greater than 0.5, a relationship between the heat flux amplitudes and the interior temperature response can be established. It is interesting to note that this approximate method requires one interior temperature response, and it does not rely upon a series formulation.

Sparrow[5] treated the inverse problem in a different manner. By Laplace transform techniques, a solution for the transformed temperature is obtained in the conventional manner. In turn, the two constants of integration may be eliminated by the thermal symmetry requirement inherent with solids, and the interior response. At this stage, the resultant temperature expression cannot be inverted; consequently a special function is introduced for this purpose. Thus, the inversion procedure is now feasible, hence the prediction temperature can be expressed as an integral equation. Unfortunately, the integral must be evaluated numerically, and several illustrative examples are presented demonstrating the procedure. Deverall[6] amends the previous method to facilitate computation of the surface heat flux in a direct manner. Nevertheless, the heat flux expression remains an integral equation requiring numerical evaluation. For a finite cylinder, Sabherwal[7] obtains a solution by successive application of the Laplace and finite Fourier transforms. The method is straightforward and the prediction temperature is obtained as an integral. It must be noted that Sabherwal does not present any mathematical justification for the inverse Laplace transform used in the paper. Since this is a key step in the development, the end result is, therefore, questionable. In a similar manner, Masket[8] develops solutions for solids subjected to various end conditions. By analyses of the direct conduction problem, an expression relating the transformed surface temperature to the interior temperature is established. Purportedly, this expression may be inverted for the desired surface temperature function. As in the preceding investigation,[7] the inversion procedure is dubious since the required transform pair does not exist. The difficulty stems from the lack of recognition that the cited functions do not satisfy the requirements of Jordan's Lemma in complex variable analysis.

In the series methods previously discussed, the success of the methods depend upon knowledge of the interior temperature and heat flux. For solids such as spheres and cylinders, thermal symmetry provides an additional boundary condition; consequently the heat flux is obtained by analysis of the direct conduction problem for the appropriate region. However, a solution is feasible for the finite slab only when the thermal symmetry is specified. Consequently, for hollow geometries and finite slabs with no inherent thermal symmetry, the series method is severely limited since the local heat flux cannot be evaluated properly. The method, in addition, relies upon the higher order derivatives of the interior temperature response. In practical applications, it is desirable to approximate the local temperature data as simply as possible, i.e., a polynomial power series. The degradation in the value of the higher derivatives due to the erosive action of successive differentiation of the power series is apparent, and, as shown in Ref. 2, a power series approximation must be rejected for uniform initial conditions. On the other hand, the transform technique for the development of the inverse conduction solution has none of the previously referred to limitations. The main difficulty, however, is in the necessity of numerical evaluation of the integral equation. If a polynomial series approximation is used for the local temperature, then several integrals must be computed numerically.

The present investigation differs from the preceding approaches, in that it develops a solution to the inverse problem in terms of readily obtainable experimental data, thermocouple readings. The method allows for a series approximation for the temperatures, and it does not require any differentiations. Since the local heat flux is no longer necessary, solutions are feasible for heat flow situations which do not possess thermal symmetry, i.e., hollow spheres, and slabs whose inner and outer surfaces are subjected to different temperatures. Based upon data from two interior thermocouple locations, a closed form solution is obtained for the prediction temperature, and extrapolation is possible beyond the thermocouple positions.

Analysis of the Linear Inverse Conduction Problem

Plane Slab or Hollow Sphere

For a solid with constant physical properties, the mathematical formulation of the two thermocouple problem is

$$(1/x^j)(\partial/\partial x)[x^j \partial T/\partial x] = (1/\alpha)\partial T/\partial t, \qquad x \geqq x_i \tag{1}$$

with the two prescribed interior conditions

$$T(x_1, t) = T_1(t) \quad \text{and} \quad T(x_2, t) = T_2(t), \quad x_2 > x_1 > x_i \tag{2}$$

where $j = 0, 1, 2$ for a slab, cylinder, or sphere, respectively, and x_i is the location of the inner face. It is desired, therefore, to find the temperatures beyond the thermocouple position, i.e., $x < x_1$ or $x > x_2$.

Starting with the slab, the Laplace transform of Eq. (1) reduces it to a second-order differential equation whose solution is readily obtained. Accordingly, under the assumption of uniform initial conditions, the transform of the reduced temperature is

$$\bar{T}(x,p) = \frac{\bar{T}_1}{1-e^{-2p\Delta}}[e^{-p(x-x_1)} - e^{-p(2\Delta+x_1-x)}] +$$
$$\left[\frac{\bar{T}_2}{1-e^{-2p\Delta}}\right][e^{-p(\Delta+x_1-x)} - e^{-p(\Delta-x_1+x)}]$$

where

$$\Delta = x_2 - x_1 \qquad \text{and} \qquad p = (s/\alpha)^{1/2} \tag{3}$$

Equation (3), as it appears, cannot be inverted due to the presence of the positive arguments in the exponents. However, by suitable selection of the form of the interior temperature $\bar{T}_1$ and $\bar{T}_2$, this difficulty may be circumvented. Since the method of solution varies slightly for the region $x < x_1$ and $x > x_2$, the analysis will be divided accordingly.

For backward extrapolation, $x < x_1$, a relationship between the thermocouples is written as

$$\bar{T}_1(p) = \bar{T}_2 \sum_{m=1}^{\infty} A_m e^{-mp\Delta} \tag{4}$$

where the temperature at x_2 is expressed as the power series

$$T_2 = \sum_{n=1}^{\infty} b_n t^n/n! \tag{5}$$

The summation term in Eq. (4) represents an enabling function whose coefficients A_m are evaluated from the temperature response at x_1. Substitution of Eq. (4) and the Laplace transform of Eq. (5) into Eq. (3), yields

$$\bar{T}(x,p) = \sum_{m=1}^{\infty} A_m \sum_{n=2}^{\infty} \frac{b_{n-1}}{s^n} \sum_{q=0,2,4}^{\infty} \times$$
$$\{e^{-p[x-x_1+(m+q)\Delta]} - e^{-p[x_1-x+(m+q+2)\Delta]}\} +$$
$$\sum_{n=2}^{\infty} \frac{b_{n-1}}{s^n} \sum_{q=1,3,5}^{\infty} \{e^{-p[x_1-x+q\Delta]} - e^{-p[x-x_1+q\Delta]}\} \tag{6}$$

where the denominator in Eq. (3) is replaced by a sum of negative exponentials. If it is now stipulated that $\Delta \geqq x_1 - x_i$, then Eq. (6) can be inverted by application of the appropriate Laplace transform pairs. Since the positioning of the thermocouples by the experimenter is flexible, this requirement can easily be incorporated in the experimental design. The expression for the prediction temperature is, therefore, for $x_i \leqq x \leqq x_1$

$$T(x,t) = \sum_{m=1}^{\infty} A_m \sum_{n=1}^{\infty} b_n \sum_{q=0,2,4}^{\infty} (4t)^n i^{2n} \times \left[\operatorname{erfc}\frac{x-x_1+(m+q)\Delta}{2(\alpha t)^{1/2}} - \operatorname{erfc}\frac{x_1-x+(m+q+2)\Delta}{2(\alpha t)^{1/2}}\right] + \sum_{n=1}^{\infty} b_n \sum_{q=1,3,5}^{\infty} (4t)^n i^{2n} \times \left[\operatorname{erfc}\frac{x_1-x+q\Delta}{2(\alpha t)^{1/2}} - \operatorname{erfc}\frac{x-x_1+q\Delta}{2(\alpha t)^{1/2}}\right], \quad \Delta \geqq x_1 - x_i \tag{7}$$

with

$$T_1(t) = \sum_{m=1}^{\infty} A_m \sum_{n=1}^{\infty} b_n (4t)^n i^{2n} \operatorname{erfc}\frac{m\Delta}{2(\alpha t)^{1/2}} \tag{8}$$

Equation (7) represents the prediction temperature for all values of $x < x_1$ based upon the two thermocouple readings T_1 and T_2, respectively. The inverse of Eq. (4) is indicated by Eq. (8) where the thermocouple trace at x_2 is replaced by a polynomial expression. The coefficients A_m, in the enabling function are obtained by fitting the thermocouple data at x_1 with the quasi-power series approximation, Eq. (8).

For forward extrapolation, $x > x_2$, an analogous relationship, for the thermocouple data, as per Eq. (4), may be written as

$$\bar{T}_2(p) = \bar{T}_1 \sum_{m=1}^{\infty} B_m e^{-mp\Delta} \tag{9}$$

where the temperature at x_1 is now approximated by the polynomial series

$$T_1 = \sum_{n=1}^{\infty} a_n t^n / n! \tag{10}$$

The transform of the prediction temperature is obtained directly from Eq. (3), after substitution of Eq. (9) and the transform of Eq. (10),

$$\bar{T}(x,p) = \sum_{m=1}^{\infty} B_m \sum_{n=2}^{\infty} \frac{a_{n-1}}{s^n} \sum_{q=1,3,5}^{\infty} \times \{e^{-p[x_1-x+(m+q)\Delta]} - e^{-p[x-x_1+(m+q)\Delta]}\} + \sum_{n=2}^{\infty} \frac{a_{n-1}}{s^n} \sum_{q=0,2,4}^{\infty} \{e^{-p[x-x_1+q\Delta]} - e^{-p[x_1-x+(2+q)\Delta]}\} \tag{11}$$

For the condition $\Delta \geqq x_1 - x_i$, Eq. (11) can be readily inverted since the transform pairs are the same as those used for Eq. (7). The prediction temperature, for $x_2 \leqq x \leqq 2\Delta + x_1$, is therefore

$$T(x,t) = \sum_{n=1}^{\infty} a_n \sum_{q=0,2,4}^{\infty} (4t)^n i^{2n} \times \left[\operatorname{erfc}\frac{x-x_1+q\Delta}{2(\alpha t)^{1/2}} - \operatorname{erfc}\frac{x_1-x+(2+q)\Delta}{2(\alpha t)^{1/2}}\right] + \sum_{m=1}^{\infty} B_m \sum_{n=1}^{\infty} a_n \sum_{q=1,3,5}^{\infty} (4t)^n i^{2n} \times \left[\operatorname{erfc}\frac{x_1-x+(m+q)\Delta}{2(\alpha t)^{1/2}} - \operatorname{erfc}\frac{x-x_1+(m+q)\Delta}{2(\alpha t)^{1/2}}\right], \quad \Delta \geqq x_1 - x_i \tag{12}$$

with

$$T_2(t) = \sum_{m=1}^{\infty} B_m \sum_{n=1}^{\infty} a_n (4t)^n i^{2n} \operatorname{erfc}\frac{m\Delta}{2(\alpha t)^{1/2}} \tag{13}$$

The coefficients B_m in the enabling function, Eq. (9), are computed in the same manner as previously described for the terms A_m. A polynomial approximation is made for T_1, and Eq. (13) is matched to the thermocouple data at x_2. It should be noted, however, that the range for forward extrapolation is limited to $x_2 \leqq x \leqq 2\Delta + x_1$. For values of $x > 2\Delta + x_1$ the arguments of the exponents in Eq. (11) become positive quantities, and the inverse transform for this condition does not exist. In the finite slab case, the thermocouples can be positioned in relation to the two faces so that the preceding requirement, $x \leqq 2\Delta + x_1$ is met. If this cannot be accomplished, i.e., a thick wall, then forward extrapolation is first carried out to the extreme value $x = 2\Delta + x_1$. The problem is then reformulated in terms of the temperatures, T_2 and the temperature at the extreme value and another forward extrapolation is accomplished. This method of extending the solution is analogous to the analytic continuation principle in complex variables.

The preceding development for the slab may also be applied to spheres, with slight modifications. For spherical geometry, Eq. (1), becomes the one dimensional heat conduction equation via the transformation, $u(x,t) = xT(x,t)$, where x, represents the radial space coordinate. The prescribed interior conditions are, therefore, rewritten as $u(x_1,t) = x_1 T_1(t)$ and $u(x_2,t) = x_2 T_2(t)$, respectively. The transform of the reduced temperature, $\bar{T}(x,p)$, is obtained in a straightforward manner from the transformed differential equation

$$\bar{T}(x,p) = \frac{\bar{T}_1}{1-e^{-2p\Delta}}\left[e^{-p(x-x_1)} - e^{-p(2\Delta+x_1-x)}\right]\left[\frac{x_1}{x}\right] + \frac{\bar{T}_2}{1-e^{-2p\Delta}}\left[e^{-p(\Delta+x_1-x)} - e^{-p(\Delta-x_1+x)}\right]\left[\frac{x_2}{x}\right] \tag{14}$$

Comparison of Eqs. (14) and (3) reveals that the equations are similar; hence, the development for the prediction temperatures for $x_i \leqq x \leqq x_1$ and $x_2 \leqq x \leqq 2\Delta + x_1$ follows in the same fashion as for the slab. Consequently, the extrapolation temperatures for the sphere are obtained from Eqs. (7) and (12) by multiplying the first bracketed term by the spatial ratio, x_1/x, and the second bracketed term by, x_2/x, respectively.

Two Component Solid

As a further demonstration of the method's applicability, consider a composite solid with a thermocouple location in each of the materials. The mathematical statement of the inverse problem is therefore

$$\begin{aligned} \partial^2 T'/\partial x^2 &= (1/\alpha_1)\,\partial T'/\partial t \qquad x_i \leqq x \leqq \bar{x} \\ \partial^2 T''/\partial x^2 &= (1/\alpha_2)\,\partial T''/\partial t \qquad x \geqq \bar{x} \end{aligned} \tag{15}$$

with the interior thermocouple requirement

$$T'(x_1,t) = T_1, \quad x_1 < \bar{x}: \qquad T''(x_2,t) = T_2, \quad x_2 > \bar{x} \tag{16}$$

and the interface condition

$$T'(\bar{x},t) = T''(\bar{x},t): \quad K_1\,\partial T'/\partial x = K_2\,\partial T''/\partial x \quad \text{at} \quad x = \bar{x} \tag{17}$$

From the transform of the partial differential equations, Eq. (15), the reduced temperatures are

$$\begin{aligned} \bar{T}'(x,p) &= C_1 e^{-p_1 x} + C_2 e^{p_1 x}, \qquad x_i \leqq x \leqq \bar{x} \\ \bar{T}''(x,p) &= C_3 e^{-p_2 x} + C_4 e^{p_2 x}, \qquad x > \bar{x} \end{aligned} \tag{18}$$

where the constants of integration are evaluated from Eqs. (16) and (17). Performing the required substitutions, and after considerable manipulation, Eq. (18) may be rewritten as

$$\bar{T}'(x,s) = \frac{\bar{T}_1}{1-e^{-4p_2\Delta/(1+k)}}\left\{e^{p_1(x_1-x)} - e^{-p_1(x_1-x-4p_2\Delta/(1+k))} - \left[\frac{1-\sigma}{1+\sigma}\right]\left[e^{p_1(x_1-x)} - e^{-p_1(x_1-x)}\right]e^{-2p_2\Delta/(1+k)}\right\} + \frac{\bar{T}_2[1+\sigma]\,e^{-2p_2\Delta/(1+k)}}{2[1-e^{-4p_2\Delta/(1+k)}]}\left\{e^{-p_1(x_1-x)} - e^{p_1(x_1-x)} + \left[\frac{1-\sigma}{1+\sigma}\right]^2\left[e^{p_1(x_1-x)} - e^{-p_1(x_1-x)}\right]\right\}, \quad x < \bar{x} \tag{19}$$

and

$$\bar{T}''(x,s) = \frac{2\bar{T}_1}{[1+\sigma][1-e^{-4p_2\Delta/(1+k)}]} \times \left\{\exp\left[\frac{p_2}{1+k}[(1+k)(x_1-x)-(1-k)\Delta]\right] - \exp\left[\frac{-p_2}{1+k}[(1+k)(x_1-x)+(3+k)\Delta]\right]\right\} + \frac{\bar{T}_2}{1-e^{-4p_2\Delta/(1+k)}}\left\{\exp[-p_2(x_1-x+\Delta)] - \exp\left[\frac{p_2}{1+k}[(1+k)(x_1-x)-(3-k)\Delta]\right] + \left[\frac{1-\sigma}{1+\sigma}\right]\left[\exp\left\{\frac{-p_2}{1+k}[(1+k)(x_1-x)+(3+k)\Delta]\right\} - \exp\left(\frac{p_2}{1+k}[(1+k)(x_1-x)-(1-k)\Delta]\right)\right]\right\}, \quad x > \bar{x} \qquad (20)$$

where $p_j = (s/\alpha_j)^{1/2}$, $j = 1,2$. The elimination procedure for the constants of integration C_n are, in the main, algebraic operations; consequently these details have been deleted. It must be noted, however, that the preceding equations are predicated upon the relationship $\bar{x} = (1+k)x_1 + k\Delta/1+k$. This qualification occurs as a consequence of the algebraic simplifications leading to Eqs. (19) and (20), and it locates the thermocouple positions relative to the interface $\bar{x}$.

For backward extrapolation $x < x_1$, the enabling function for the thermocouple response is

$$\bar{T}_1 = \bar{T}_2 \sum_{m=1}^{\infty} A_m e^{-2mp_2\Delta/1+k} \qquad (21)$$

and by the same token the appropriate function for forward extrapolation $x > x_2$ is

$$\bar{T}_2 = \bar{T}_1 \sum_{m=1}^{\infty} B_m e^{-2mp_2\Delta/1+k} \qquad (22)$$

Consequently, the prediction temperature, for $x_i \leqq x \leqq x_1$, is by application of the transform pairs

$$T'(x,t) = \sum_{m=1}^{\infty} A_m \sum_{n=1}^{\infty} b_n \sum_{q=0}^{\infty} (4t)^n i^{2n} \times \left\{\operatorname{erfc}\left[\frac{2\Delta(m+2q)/(1+k)+(x-x_1)/k}{2(\alpha_2 t)^{1/2}}\right] - \operatorname{erfc}\left[\frac{2\Delta(m+2q+2)/(1+k)+(x_1-x)/k}{2(\alpha_2 t)^{1/2}}\right] - \left(\frac{1-\sigma}{1+\sigma}\right)\left(\operatorname{erfc}\left[\frac{2\Delta(m+2q+1)/(1+k)+(x-x_1)/k}{2(\alpha_2 t)^{1/2}}\right] - \operatorname{erfc}\left[\frac{2\Delta(m+2q+1)/(1+k)+(x_1+x)/k}{2(\alpha_2 t)^{1/2}}\right]\right)\right\} + \frac{1+\sigma}{2}\sum_{n=1}^{\infty} b_n \sum_{q=1,3,5}^{\infty} (4t)^n i^{2n} \times \left\{\operatorname{erfc}\left[\frac{2q\Delta/(1+k)+(x_1-x)/k}{2(\alpha_2 t)^{1/2}}\right] - \operatorname{erfc}\left[\frac{2q\Delta/(1+k)+(x-x_1)/k}{2(\alpha_2 t)^{1/2}}\right] + \left(\frac{1-\sigma}{1+\sigma}\right)^2 \times \left(\operatorname{erfc}\left[\frac{2q\Delta/(1+k)+(x-x_1)/k}{2(\alpha_2 t)^{1/2}}\right] - \operatorname{erfc}\left[\frac{2q\Delta/(1+k)+(x_1-x)/k}{2(\alpha_2 t)^{1/2}}\right]\right)\right\}$$

$$\Delta \geqq (1+k)x_1 - 2kx_i/2k$$

with

$$T_1(t) = \sum_{m=1}^{\infty} A_m \sum_{n=1}^{\infty} b_n (4t)^n i^{2n} \operatorname{erfc}\frac{m\Delta}{(1+k)(\alpha_2 t)^{1/2}} \qquad (23)$$

Similarly, the prediction temperature for $x_2 \leqq x \leqq (3+k)\Delta + (1+k)(x_1)/1+k$ is

$$T''(x,t) = \frac{2}{1+\sigma}\sum_{n=1}^{\infty} a_n \sum_{q=0}^{\infty} (4t)^n i^{2n} \times \left\{\operatorname{erfc}\left[\frac{x-x_1+[(1+4q-k)/(1+k)]\Delta}{2(\alpha_2 t)^{1/2}}\right] - \operatorname{erfc}\left[\frac{x_1-x+[(3+4q+k)/(1+k)]\Delta}{2(\alpha_2 t)^{1/2}}\right]\right\} + \sum_{m=1}^{\infty} B_m \sum_{n=1}^{\infty} a_n \sum_{q=0}^{\infty} (4t)^n i^{2n} \times \left\{\operatorname{erfc}\left[\frac{x_1-x+\Delta+[(2m+4q)/(1+k)]\Delta}{2(\alpha_2 t)^{1/2}}\right] - \operatorname{erfc}\left[\frac{x-x_1+[(3+2m+4q-k)/(1+k)]\Delta}{2(\alpha_2 t)^{1/2}}\right] + \left(\frac{1-\sigma}{1+\sigma}\right)\left(\operatorname{erfc}\left[\frac{x_1-x+[(3+2m+4q+k)/(1+k)]\Delta}{2(\alpha_2 t)^{1/2}}\right] - \operatorname{erfc}\left[\frac{x-x_1+[(1+2m+4q-k)/(1+k)]\Delta}{2(\alpha_2 t)^{1/2}}\right]\right)\right\},$$

$$\Delta \geqq (1+k)x_1 - 2kx_i/2k$$

with

$$T_2(t) = \sum_{m=1}^{\infty} B_m \sum_{n=1}^{\infty} a_n (4t)^n i^{2n} \operatorname{erfc}\frac{m\Delta}{(1+k)(\alpha_2 t)^{1/2}} \qquad (24)$$

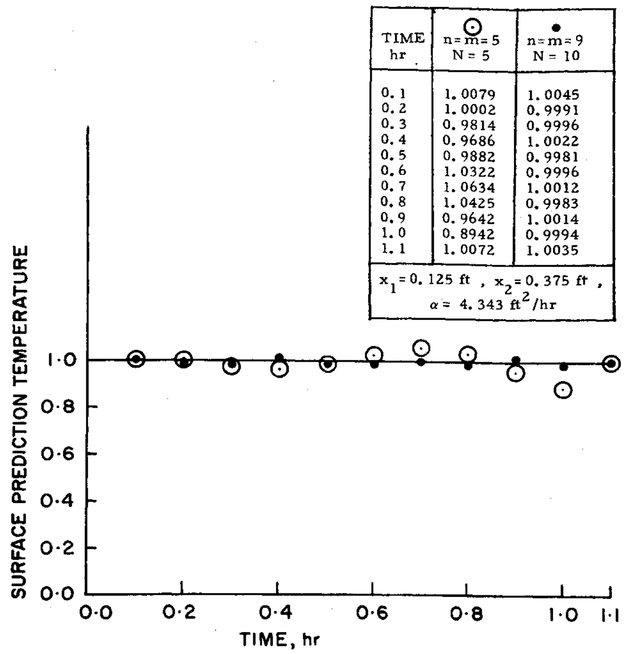
The preceding prediction equation may, therefore, be applied when the thermocouples are positioned in accordance with the requirements: $\bar{x} = (1+k)x_1 + k\Delta/1+k$ and $\Delta \geqq (1+k)x_1 - 2kx_i/2k$. Both of these requirements enable Eqs. (19) and (20) to be inverted for the extended ranges in spatial variables x. In the forward extrapolation scheme, there is an added requirement, $x_2 \leqq x \leqq (3+k)\Delta + (1+k)x_1/1+k$, the rationale for this follows from the discussion appearing after Eq. (13).

The solution method derived for the two component solid may be extended to include the multicomponent case. Consider a wall with many layers of different material, in which, the first two layers have thermocouples positioned, one in each layer. From Eq. (23), the prediction temperatures are obtained in the first layer. Obviously, the differing thermal properties will determine the values of $\bar{x}$ and Δ to be associated with this solution. If, on the other hand, thermocouples are located in the last two layers, then Eq. (24) determines the temperature in the last layer and different values of $\bar{x}$ and Δ ensue. The inverse conduction problem for the multicomponent wall is, therefore, tractable when data from four thermocouples is available.

Discussion of Results

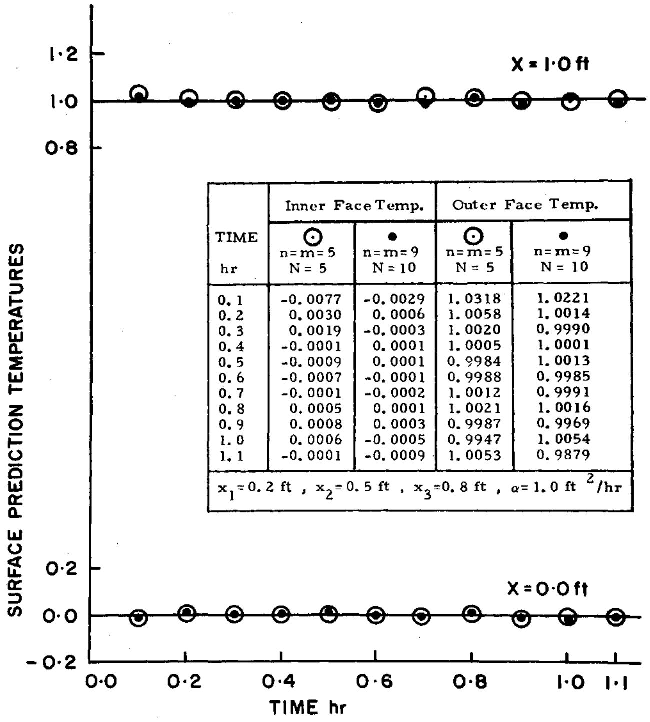
The principles of the inversion procedure, as derived in the preceding section, were applied to several situations. The graphical results for three cases are reported in the present study, and they are shown in Figs. 1–4. Since the computation method is the same for all the cases considered, it is only necessary to present the details of the inversion method for a representative problem. In what follows, the inversion method is applied to a semi-infinite solid initially at zero temperature and whose surface is raised to unity.

From the analytical solution to the direct conduction problem the temperature traces at the interior spatial positions x_1 and x_2 are generated. This information is analogous to the thermocouple data that would normally be obtained experimentally. For backward extrapolation, the thermocouple response at x_2 is approximated by a nth degree polynomial over a time range which has been divided into N equal parts. In turn, the temperature data for discrete values of time at x_1 is substituted into Eq. (8), and the values of the enabling function coefficients B_m are established. The data at both locations is fitted by the



TIME hr	⊙ n=m=5 N = 5	• n=m=9 N = 10
0.1	1.0079	1.0045
0.2	1.0002	0.9991
0.3	0.9814	0.9996
0.4	0.9686	1.0022
0.5	0.9882	0.9981
0.6	1.0322	0.9996
0.7	1.0634	1.0012
0.8	1.0425	0.9983
0.9	0.9642	1.0014
1.0	0.8942	0.9994
1.1	1.0072	1.0035
$x_1 = 0.125$ ft, $x_2 = 0.375$ ft, $\alpha = 4.343$ ft^2/hr		

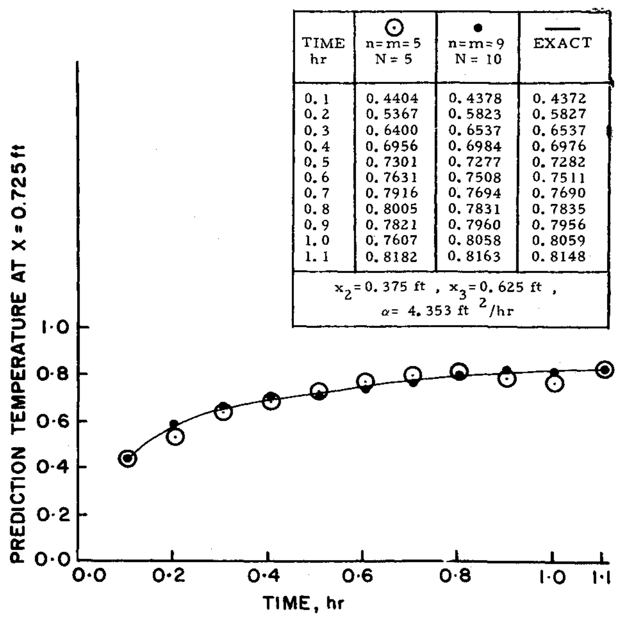
Fig. 1 Results of the surface temperature, Eq. (7), for a semi-infinite solid whose surface is raised to unity.



TIME hr	Inner Face Temp.		Outer Face Temp.	
	⊙ n=m=5 N = 5	• n=m=9 N = 10	⊙ n=m=5 N = 5	• n=m=9 N = 10
0.1	-0.0077	-0.0029	1.0318	1.0221
0.2	0.0030	0.0006	1.0058	1.0014
0.3	0.0019	-0.0003	1.0020	0.9990
0.4	-0.0001	0.0001	1.0005	1.0001
0.5	-0.0009	0.0001	0.9984	1.0013
0.6	-0.0007	-0.0001	0.9988	0.9985
0.7	-0.0001	-0.0002	1.0012	0.9991
0.8	0.0005	0.0001	1.0021	1.0016
0.9	0.0008	0.0003	0.9987	0.9969
1.0	0.0006	-0.0005	0.9947	1.0054
1.1	-0.0001	-0.0009	1.0053	0.9879
$x_1 = 0.2$ ft, $x_2 = 0.5$ ft, $x_3 = 0.8$ ft, $\alpha = 1.0$ ft^2/hr				

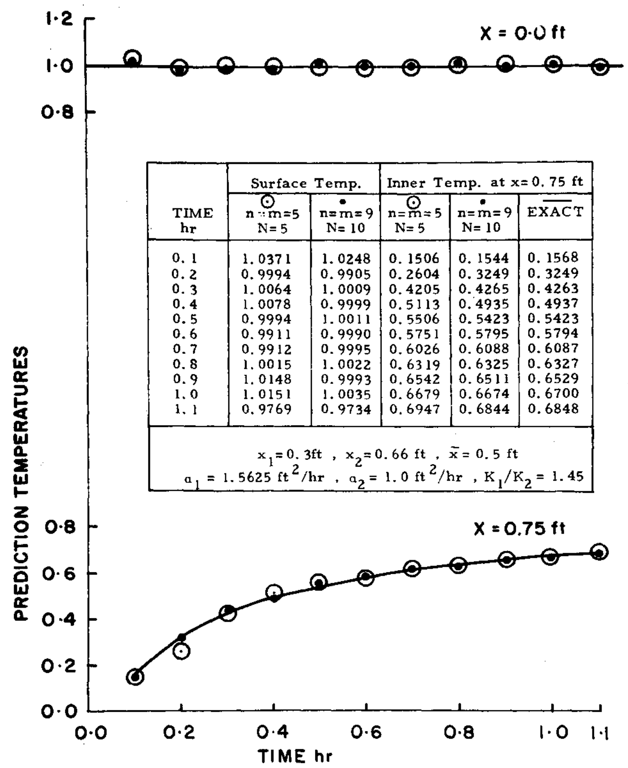
Fig. 3 Surface prediction temperatures for a finite slab of unit thickness.

least square technique. It should be noted that deviations of less than 1% were customarily observed when Eq. (8) was compared with the entire temperature trace at x_1. As shown in Fig. 1, the surface prediction temperature, Eq. (7), is obtained for a semi-infinite solid composed of copper. The thermocouples were positioned at $x_1 = 0.125$ ft and $x_2 = 0.375$ ft, and the graphs indicate the results of the inversion method. It is apparent that as the power of the approximation polynomial, $n = m$, increases, the accuracy of the method improves. Correspondingly, the time range from 0.1 hr to 1.1 hr under consideration is divided into smaller equal parts. This effect is noted on the graphs as $N = 5$ and $N = 10$. The forward extrapolation temperature may be obtained from Eq. (12) where the temperature traces at x_1 and x_2 are approximated by Eqs. (10) and (9), respectively. In an experimental program this procedure would be the only method available for forward temperature predictions. There is however, an alternative procedure applicable to the illustrative example by which the computation time may be reduced considerably. A third temperature trace is computed from the analytical solution to the right of the position x_2, i.e., x_3, and in this manner, the coefficients in the power series previously generated at position x_2 for backward extrapolation may be used again. Equation (13) is,



TIME hr	⊙ n=m=5 N = 5	• n=m=9 N = 10	—— EXACT
0.1	0.4404	0.4378	0.4372
0.2	0.5367	0.5823	0.5827
0.3	0.6400	0.6537	0.6537
0.4	0.6956	0.6984	0.6976
0.5	0.7301	0.7277	0.7282
0.6	0.7631	0.7508	0.7511
0.7	0.7916	0.7694	0.7690
0.8	0.8005	0.7831	0.7835
0.9	0.7821	0.7960	0.7956
1.0	0.7607	0.8058	0.8059
1.1	0.8182	0.8163	0.8148
$x_2 = 0.375$ ft, $x_3 = 0.625$ ft, $\alpha = 4.353$ ft^2/hr			

Fig. 2 Results of the prediction temperature, Eq. (12), for the semi-infinite solid.



TIME hr	Surface Temp.		Inner Temp. at x = 0.75 ft		
	⊙ n=m=5 N=5	• n=m=9 N=10	⊙ n=m=5 N=5	• n=m=9 N=10	—— EXACT
0.1	1.0371	1.0248	0.1506	0.1544	0.1568
0.2	0.9994	0.9905	0.2604	0.3249	0.3249
0.3	1.0064	1.0009	0.4205	0.4265	0.4263
0.4	1.0078	0.9999	0.5113	0.4935	0.4937
0.5	0.9994	1.0011	0.5506	0.5423	0.5423
0.6	0.9911	0.9990	0.5751	0.5795	0.5794
0.7	0.9912	0.9995	0.6026	0.6088	0.6087
0.8	1.0015	1.0022	0.6319	0.6325	0.6327
0.9	1.0148	0.9993	0.6542	0.6511	0.6529
1.0	1.0151	1.0035	0.6679	0.6674	0.6700
1.1	0.9769	0.9734	0.6947	0.6844	0.6848
$x_1 = 0.3$ ft, $x_2 = 0.66$ ft, $\tilde{x} = 0.5$ ft, $\alpha_1 = 1.5625$ ft^2/hr, $\alpha_2 = 1.0$ ft^2/hr, $K_1/K_2 = 1.45$					

Fig. 4 Prediction temperatures for a composite solid.

therefore, fitted to the temperature data at x_3. The same technique can be utilized in any experimental program incorporating three thermocouples. The results for this situation are shown in Fig. 2, where the third thermocouple is located at $x_3 = 0.625$ ft, and the temperature predicted at $x = 0.725$ ft.

As further illustration of the inversion method's accuracy, solutions were developed for a slab of unit thickness, initially at zero temperature, whose inner and outer surfaces are maintained at zero and unit temperatures, respectively. For the selected thermocouple locations indicated in Fig. 3, the prediction temperatures are computed in the same manner as described for the semi-infinite solid.

Lastly, the numerical results for the composite solid are presented graphically in Fig. 4. The particular geometry selected was that of a finite slab, 0.5 ft thick, backed by a semi-infinite solid with different thermal properties. In this case, the thermal properties, K_1 and α_1, refer to the first layer of material, and the terms K_2 and α_2 are associated with the second, the semi-infinite solid. Assuming a zero initial temperature, the composite solid thus described is heated via a unit step change in its surface temperature. From Eq. (23), the backward extrapolation of the temperature at the surface is computed when the two thermocouples are positioned at $x_1 = 0.3$ ft and $x_2 = 0.66$ ft. Similarly, the forward extrapolation of the temperature to $x = 0.75$ ft is obtained from Eq. (24). It should be noted that in this illustrative example, only the thermocouple data at x_1 and x_2 are utilized for the forward temperature projection.

Conclusions

An analytical solution has been developed for the inverse conduction problem which utilizes the temperature data from two interior thermocouples. As presented in the analysis section for a single component solid, the distances between the thermocouples must satisfy the requirement $\Delta \geqq x_1 - x_i$. In an experimental program where the positioning of the thermocouples is flexible, the preceding restriction can be easily satisfied. A generalized method is also presented to treat multicomponent solids. For either situation, the temperature may be extrapolated beyond the thermocouple locations. Since the resultant expressions for the temperature involve tabulated functions, i.e., repeated integrals of the error function, numerical evaluation of the prediction temperatures presents little difficulty.

Inspection of the computational results in Figs. 1–4 illustrate the high degree of accuracy that can be achieved. By increasing the power of the approximation polynomials, improved accuracy results. It should be noted, however, that as n increases, the number of terms in the series summation also increases, thereby lengthening the computer time. The three numerical examples were chosen to demonstrate the applicability of the method to geometries with no thermal symmetry. As an additional test of the method, the stepwise change in one of the surface temperatures was also assumed. It was felt that this would be a severe test of the method due to resultant steep temperature gradients.

The solution for the hollow sphere is obtained in a straightforward manner from Eqs. (7) and (12). Numerical computations were performed for a hollow sphere subjected to the same boundary conditions as applied to the preceding case of the slab. The outer surface experienced a stepwise change in temperature while the inner radius was maintained at zero temperature. It is sufficient to report that the method successfully predicted the surface temperatures.

Obviously, the two thermocouple solution is not limited to one directional heat flow situations which do not possess thermal symmetry. It can be utilized for a solid sphere or insulated slab as well. Numerical results were obtained for test cases shown in Refs. 2 and 5. The prediction temperatures matched the imposed surface temperatures.‡

In conclusion, a closed form solution to the inverse conduction problem is obtained. The method does not require evaluation of the higher order time derivatives of a local temperature or heat flux, or numerical integration of complex integrals. By judicious thermocouple placement, the temperatures may be extrapolated beyond the range of the thermocouples, in either direction. As demonstrated in several illustrative examples, a high degree of accuracy can be achieved.

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‡ For brevity's sake, the graphical results for the sphere and insulated slab are not presented at this time, see Ref. 9.